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From Dürer's Magic Square to Klumpenhouwer Tesseracts: On *Melencolia* (2013) by Philippe Manoury

Magic squares stand as some of the most popular objects in recreational mathematics. They probably first appeared in China – the earliest unambiguous reference dates from the first century CE (Cammann 1960, pp. 116–18). Although their origins were intrinsically rooted in religious and esoteric contexts, magic squares gradually became an autonomous mathematical entity. In particular, several Arabic treatises from the Middle Ages propounded fairly accurate algorithms which demonstrated how to build them (Sesiano 2017). These sources probably had an impact in Europe during the Renaissance, where they penetrated by way of Andalusia (Comes 2016), and later in the specific field of compositional practices.

As the history of Western music reveals, many musicians of different periods, including those living in recent times, have allowed mysticism and occultism to nourish their theoretical conceptions. It is not surprising, therefore, that some of these musicians have dealt with magic squares for esoteric purposes (Hasler 2007). Conversely, many other composers have regarded tabulated data as a device which could help them clarify and objectify their musical practice. The source of this readiness is partly to be found in the research in combinatorics and music which arose in Europe during the seventeenth and eighteenth centuries (Knobloch 2001). To these combinatorial devices belongs dice music, which was mainly an amusement for dilettantes during the second half of the eighteenth century (Hedges 1978). Tabulated diagrams became less anecdotal, however, with the end of the common practice period, after which composers sought new methods to stimulate their creative thoughts. Latin and magic squares, as well as other related numerical constructions, are therefore found in several composers' sketches of the twentieth and twenty-first centuries. A representative but non-exhaustive list of European composers includes Anton Webern, Ernst Krenek, Olivier Messiaen, Bruno Maderna, Pierre Boulez, Michel Philippot, Peter Maxwell Davies, Hans Zender and Walter Zimmermann (for an account, see Besada and Andreatta 2021).

Our own research follows in the steps of the analysts who studied the works of these composers. We will focus here on Philippe Manoury's Third String Quartet – with crotales – *Melencolia (d'après Dürer)* (2013), a work inspired by Albrecht Dürer's engraving of the same name (Plate 1). Close to its upper right corner, the artist included a magic square – henceforth Dürer's

Plate 1 Albrecht Dürer, *Melencolia I* [Colour figure can be viewed at wileyonlinelibrary.com]

square – which displays the date of the artwork (1514) in the middle of its bottom row. The square is also reproduced in the prologue of Manoury's score, a fact which suggests a much deeper relationship between the engraving and the music than simply a paratextual (Genette 1982) connection of the titles. As our analyses will demonstrate, Manoury's use of Dürer's square could be regarded as a particular case of musical ekphrasis (Goehr 2010), following a large tradition of musicians' comments on paintings, from the past to the twentieth and twenty-first centuries (Bruhn 2000 and 2004).

Among Manoury's sketches of the formal structure of *Melencolia*, there is a document (Plate 2) which labels three basic categories, namely *carré magique*

Plate 2 Detail of Manoury's formal planning for *Melencolia*. Reproduced by kind permission of the author

(magic square), *grammaire* (grammar) and *échelles* (scales).[1] Our analyses will focus exclusively on the musical outcomes derived from the magic square categories, that is, Dürer's square and the several modifications carried out on it by the composer. Manoury mostly used the tabulated data for generating collections of pitch classes – often tetrads – and sometimes exploited it for other purposes, such as deriving rhythmic patterns. In comparison, Manoury's logic in dealing with grammars and scales was quite dissimilar in terms of formalism and musical result. So, we have decided to set aside specific comments on these latter categories – although the grammars stand as an important technical feature of his recent music (Manoury 2012 and 2013) – and instead to concentrate on a discussion of the pitches explicitly related to Manoury's tabulated numbers.[2] We acknowledge however that, although the three categories mentioned are first presented in the score as juxtaposed entities over time, they are sometimes merged, and their perceptual implications are far from straightforward.

Our argument is structured as follows. First, we will provide a set of mathematical definitions and properties of certain numerical squares, which are necessary for our theoretical approach and analysis. Second, we will discuss the initial reasons which lead us to consider K-nets – and some extensions of these – as the most suitable analytical tool for reaching our goal. Our discussion will touch on the issue of Perle–Lansky cycles: important resemblances with our case study will partially orientate our analytical steps. Third, we will dive into the actual analysis of several passages of *Melencolia* which involve the magic square category, drawing upon Manoury's sketch material. Our analysis will follow a cumulative complexification process by looking deeply into three particular passages which rely on the aforementioned category. They will demonstrate that, despite the compositional surface diversity, there are strong structural relationships leading to formal unity. Fourth, we will step back and provide an abstract summary of the main features brought to the fore by our analysis

Fig. 1 Dürer's magic square

while aiming to enlarge their potential scope. In doing so, we will also suggest possible compositional extensions of the techniques under scrutiny. Finally, we will introduce a second level of abstraction with an eye to re-evaluating the pertinence of transformational tools for our case study.

Latin, Magic and Gnomon Squares: Definitions and Properties

A *squared array*, or more simply a *square*, is an array with the same number of rows and columns. If the expression is prefixed by a positive integer, it determines the *size* of the square, that is, its number of rows and columns. Three types of squares with specific properties are of particular significance for our analysis, namely Latin squares, magic squares and gnomon squares.

A *Latin n-square* is an *n*-square in which *n* different symbols occur *n* times but only once per row and per column. Sudokus are the most popular Latin 9-squares nowadays: in this puzzle, numbers from 1 to 9 have to be distributed across the rows and columns with no repetition. The term was coined by Leonhard Euler, who provided a constructive method for deriving magic squares from Latin squares (Dénes and Keedwell 1991, p. 3).

A *magic n-square*, with *n* greater than or equal to 3, is an *n*-square exhibiting two properties. First, it is filled with n^2 different positive integers – in other words, none of these numbers is repeated. Second, the sum of integers for each row, each column and both main diagonals is a constant, which is known as the *magic sum*. In particular, a *pure magic n-square* is a magic *n*-square wherein the positive integers take consecutive values ranging from 1 to n^2. Dürer's square – although equivalent distributions are found in China, from at least the thirteenth century (Bača and Miller 2008, pp. 23–4) – is the most famous pure magic 4-square (see Fig. 1, where our representation is bicoloured to highlight some of the properties described below). It arranges integers from 1 to 16, and its magic sum is 34.

Table 1 Summary of the main formal properties of Dürer's square

Origin	Label	Summary
magic square	magic sum	Rows, columns, and the main diagonals sum to 34.
gnomon magic square	magic semi-sum	Skewing cells sum to 17.
	additive invariance	Adjacent black cells sum to 19, white ones sum to 15.
		Analogous sums for vertical pairs are 21 and 13.
	subtractive invariance	For each row, the absolute difference of extreme cells is 3; 1 in the case of central cells.
		Analogous differences for columns are 12 and 4.

Among the possible ways of distributing numbers in a pure magic 4-square, Dürer's choice is quite singular from a mathematical viewpoint. It is actually a *gnomon magic square*, that is, a magic 4-square wherein the magic sum also appears in each of the four 2-subsquares of its corners, as well as the central 2-subsquare (Hunter and Madachy 1975, p. 24). Further geometrical patterns of the gnomon magic square also verify its magic sum (Pickover 2002, p. 21).

Dürer's square has other properties which emanate from its gnomon structure and concern the relationships between pairs of cells. First, the sum of every pair of *skewing cells* – those which are symmetrical with respect to the centre of the square – is always 17, that is, half of the magic sum. This can be checked for instance by adding the numbers from its opposite corners: $16 + 1 = 13 + 4 = 17$. Second, several sums of adjacent cells remain invariant. Let us consider horizontal pairs as depicted in our figure: adjacent entries in black always sum to 19 whereas those in white sum to 15.[3] Likewise, if we take vertical pairs, their partial sums equal 21 and 13. Third, for every row i of the 4-square, the absolute differences between extreme or central pairs of cells remain constant: $|x_{i,1} - x_{i,4}| = 3$ and $|x_{i,2} - x_{i,3}| = 3$. This can be checked for instance with the first row: $|16 - 13| = 3$ and $|3 - 2| = 1$. The same property applies to columns, taking values 12 and 4.[4]

The properties listed above are important for our analytical purposes. Some of these properties have implications regarding the distribution of pitches – and possibly the rhythmic patterns – which we will examine. There are also properties that affect the relationships among different pitches from a transformational viewpoint. Therefore, we have decided to end this section with an abridged statement of the square properties (Table 1). We suggest some 'catchy' labels which may help the reader to quickly remember them when referred to in the course of our analysis.

Entering the Transformational Debate

In our introduction, we mentioned several European composers who occasionally, whether consciously or not, used Latin or magic squares in their composition process. American musicians have also dealt with such objects for composing as well as for various analytical purposes. For example, Milton Babbitt built squared arrays to analyse Schoenberg's Violin Concerto, Op. 36 (1934–6; reproduced in Perle 1963, p. 121) or discussing Stravinsky's *Movements* (1958–9; Babbitt 1987, p. 18). Both diagrams are Latin 12-squares, a type of array already identified by scholars in the study of Babbitt's own music (e.g. Cohn 1982).

In his writings, Babbitt often included arrays to help the reader better grasp his theoretical argumentation. This choice was particularly appropriate in the discussion of invariance, which was one of his favourite topics in relation to twelve-note music (Babbitt 1960). In this context, invariance means that, although a musical entity becomes a new entity when subjected to some compositional technique, some aspects of its structure are preserved. For example, consider any of the Latin 12-squares which respectively display the transpositions and inversions of a twelve-note row. If the whole array is subjected to a non-trivial transposition, an inversion or a combination of both, the outcome will be a different square, yet also a Latin square.[5] Invariance is an important topic beyond twelve-note music analysis, as in the exhaustive list of pitch-class sets (Forte 1973, pp. 179–81). Each set stands for a collection of elements which preserve interval structure under transposition and/or inversion, such as all the major and minor triads, which are represented by Forte's prime form 3–11 [0, 3, 7].

Both Babbitt and Forte focused on the invariance of musical entities under some compositional technique. This viewpoint prioritises the objects while placing the transformation in the background. As mathematicians do when they explore duality, this priority can be swapped. This idea is precisely that which is at the core of transformational analysis in music (Lewin 1987). Among the multiple tools created from this perspective, Henry Klumpenhouwer, along with David Lewin, developed a technique which proved particularly suitable for approaching atonal music. The main interest of Klumpenhouwer networks (K-nets) is to connect different pitch-class sets by detecting stable transpositions and inversions within their respective internal structures (Lewin 1990 and 1994; and Klumpenhouwer 1991, 1994 and 1998). In other words, K-nets prioritise the invariance of transformations – and that of hyper-transformations – while relegating the more elemental structure of pitch-class sets to the background.

A few years after its publication, Lewin's first article on K-nets – which appeared in *Music Theory Spectrum* (Lewin 1990) – received a response in the same journal from George Perle. The composer's letter (Perle 1993) highlighted the intimate resemblances between these networks and the Perle–Lansky cycles he had started to explore in the 1960s. A crucial influence on his path was Alban

Fig. 2 Sequence of sum tetrachords (adapted from Perle [1977] 1996, p. 53). Accidentals affect only their related notehead

Fig. 3 Strong isographies among Perle's sum tetrachords; partial transcription
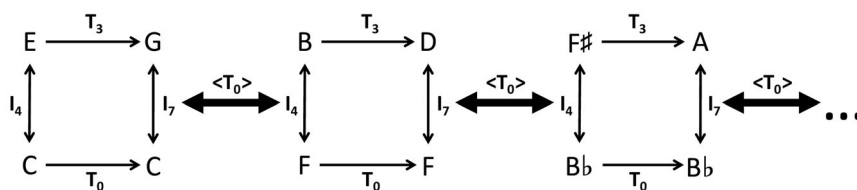
Berg's master array for transforming twelve-note rows (Perle 1977, p. 5). Perle had already published some of his theoretical results before Lewin's article (Perle 1964 and [1977] 1996). The music-theory community acknowledged Perle's remarks about K-nets, and some years later, the journal again dedicated a special issue to this topic (Headlam 2002, Lambert 2002, Lewin 2002 and Stoecker 2002). Other scholars explored these resemblances further with great subtlety (Foley 2002 and 2009).

Among the wealth of arrays and musical examples found in Perle's writings, the concept of sum tetrachords (Perle [1977] 1996, pp. 53–6) is particularly relevant to understanding Manoury's use of Dürer's square and derived combinatorial objects to generate related collections of pitch classes. Perle's main example is a particular chord sequence (reproduced in Fig. 2). We propose to partially translate it into K-nets: as Perle detected in the 1990s, his sums match with inversion operators in these networks, and his int[ervals] with transposition operators (Fig. 3). In addition, we can easily rewrite the underlying Perle–Lansky cycles as an array akin to our previous representation of Dürer's square (Fig. 4).

The redundant representation of the two bottom pitch classes in each network of Fig. 3 is an unusual unfolding of K-nets. In fact, Perle's sum tetrachords

Fig. 4 Underlying numerical array for the sequence of sum tetrachords (adapted from Perle [1977] 1996, p. 53)

0	5	10	3	8	1	6	11	4	9	2	7
7	2	9	4	11	6	1	8	3	10	5	0
4	11	6	1	8	3	10	5	0	7	2	9
0	5	10	3	8	1	6	11	4	9	2	7

are actually triadic sets; his fifth and eleventh tetrachords are even only dyads, containing pitch classes G♯ and B, and D and F, respectively. Far from being useless to us, the repetition of pitch classes within these networks will help us gain a deeper understanding of Manoury's compositional constructions. Our approach is motivated by recent developments in the theory of K-nets, in particular the poly-K-Nets which generalise the existing model and include the possibility of redundancies (Popoff, Andreatta and Ehresmann 2015 and 2018; and Popoff, Agon, Andreatta and Ehresmann 2016). In addition to this clarification, the main result of our diagrammatic interpretation is the persistence of strong isographies through different networks; this aspect is fully consistent with previous research intersecting Perle–Lansky cycles and K-nets.

As to the array which displays the data from all the interacting cycles (Fig. 4), it is not a square and its elements appear more than once. Therefore, it cannot be compared straightforwardly with Dürer's square. Despite this obvious fact, the columns of this array exhibit remarkable properties which strongly recall gnomon magic squares. First, there are only three possible sums for each column: 11, 23 and 35. Second, all the black cells of a single column sum to either 7 or 19, and the white ones sum to 4 or 16. Third, the difference of outer entries in each column is always 0, whereas the difference of inner ones is either 3 or 9 in absolute value. To provide a musically relevant interpretation of these numbers, we simply have to consider octave equivalence and reduce them to modulo 12. This choice enables us to consider numbers 11, 23 and 35 as one same value – likewise with 7 and 19 or 4 and 16 – whereas 3 and 9 are related via inversion, which leads us to uncover some structural attributes. Indeed, the columns of the array have, modulo 12, three properties analogous to those of Dürer's square: magic sum, additive invariance, and subtractive invariance. Even more important for us, two of these properties are projected onto the transformational diagrams: the additive invariance matches with the transposition operators and the subtractive invariance with the inversion ones.

Perle was intentionally working with these kinds of numerical invariances although, to our knowledge, he never mentioned magic or gnomon magic squares in his theoretical writings. Some decades later, Manoury was using Dürer's square for the composition of *Melencolia*, but was probably unaware of these previous music-theoretical constructions. This serendipity, if we may

call it that, leads us to adopt transformational tools in our following analysis of Manoury's String Quartet. We think it preferable to use K-nets instead of Perle-like arrays because Manoury incorporated other compositional criteria which may go unnoticed by taking the second option. Furthermore, the structural plasticity of network drawings plays in our favour. We aim to metamorphose the 'molecular' metaphor of K-nets (O'Donnell 1998) into a 'hypercubic' one. Thus, squared networks for tetrads will become the vertices of squared isographies which will expand first as cubes, then as tesseracts. We believe for example that cubic K-nets are handy visually speaking (e.g. Mazzola and Andreatta 2006, p. 99), and a multi-dimensional gallery of squared isographies would be an elegant offshoot from its root – Dürer's square – in terms of geometrical unfolding.

There are of course other voice-leading models for connecting the 29 classes of T/I-equivalent tetrads (e.g. Straus 2003 and Cohn 2003), models which we have carefully examined before embracing the K-net machinery. They fail to yield adequate results in the context which interests us here for two reasons: they ignore the properties of squares which we have already identified as a promising path, and they cannot account for the issue of defective sets – that is, repetition of pitch classes – whereas our model, as well as the generalised poly-K-nets, does. But the best way to convince the reader of the validity of our approach is to show our modelling of Manoury's compositional choices with K-nets.

Melodic Tetrads Generated by Dürer's Square

The opening of *Melencolia* is made of high but soft sustained pitches. The score starts with an A_6 – we consider the middle C = C_4 – in the viola; this pitch reappears, like a pedal across a long period of time (Ex. 1): it comes back in the cello in bar 3 before being picked up by violin I, and so on. In Manoury's sketch for these bars (Plate 3), the first occurrence of A_6 is labelled as *axe*, meaning axis in French. In the sketch, we also find a polygonal line connecting numbers (16, 3, 2, 13) – that is, the first row of Dürer's square – and another connecting its retrograde (13, 2, 3, 16). Although it may seem unclear from the handwritten document, a comparison of both diagrams with the first bar of *Melencolia* allows us to deduce the following equivalences: on one hand, the numerical sequence (16, 3, 2, 13) is mapped, from violin I to violin II and back to violin I, as the pitch series (C_8, B_6, $B\flat_6$, A_7); on the other hand, sequence (13, 2, 3, 16) is mapped, from cello to violin II to viola, as (A_5, $G\sharp_6$, $G\natural_6$, $F\sharp_6$).

The term *axe* provides the key to understanding these mappings. Considering the number of semitones between the A_6 pedal point and each pitch of these melodic sequences, the mechanism is unveiled: Manoury mirrored, respectively up and down, each value of the numerical sequence, minus one semitone – he started by counting 1 for the pedal pitch itself. Only the $F\sharp_6$ does not match with this protocol; it belongs to the expected pitch class but an octave higher.

Ex. 1 *Melencolia*, bars 1–4. © Durand Editions, 2013. Reproduced by kind permission of Universal Music Publishing and Durand Editions


Plate 3 Manoury's draft for the opening bars of *Melencolia*. Reproduced by kind permission of the author

The same compositional logic applies to bar 3, where pitches are derived from the second row of Dürer's square. This process continues, spanning bars 5–7. In short, Manoury conceived a protocol for mapping the numerical entries of the magic square into pitches. More accurately, as octave displacements are sometimes found, the square is mapped into pitch classes.

Fig. 5 Different tetrads generated from Dürer's square. Each bar displays both the original form and its inversion as vertical chords ignoring the actual octaves. The number below each tetrad serves as a label for subsequent discussion. This style of notation applies to further figures in this essay
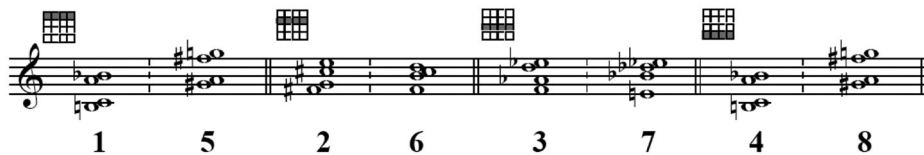
A listener accustomed to atonal music may notice many of the symmetries induced because of the recurrent parallel and contrary motions within the opening bars. Beyond this, and disregarding rhythm, we may regard the eight melodic lines derived from Dürer's square as pitch-class sets (Fig. 5). We have associated these sets in pairs – 1 and 5, 2 and 6, 3 and 7, 4 and 8 – to highlight the most evident relationship. Each pair is generated by a single row of the square. As pitch class A acts as a pitch-class mirror, it induces an I_6 relationship between the tetrads of each pair. A closer look allows us to detect further connections. The eight tetrads belong to only two of Forte's categories: tetrads 1 and 5, and 4 and 8 – which are identical to the former – are T/I-related to prime form 4–1 [0, 1, 2, 3], whereas tetrads 2, 3, 6 and 7 are T/I-related to prime form 4–13 [0, 1, 3, 6].

These features are interesting enough, analytically speaking, but we previously promised to deliver K-nets. There are of course many possibilities for organising their arrows; among them, we are going to opt for a path inspired by our previous remarks about Perle's method. As highlighted above, Dürer's square – its rows in particular – and the columns of Perle's array for sum tetrachords share three properties modulo 12: magic sum, additive invariance and subtractive invariance. Let us therefore imitate the K-nets we built in Fig. 3, which captured two of these properties, but this time with Manoury's pitches (Fig. 6). For an accurate analogy, we have to imitate the previous geometrical distribution wherein the vertices are the pitch classes and the arrows are the sides of a square. As in Perle's example, we can start by drawing the horizontal arrows, directed from left to right, corresponding to the transposition operators, and the vertical, double-headed arrows of the inversion operators. Next, again as with Perle, the subtractive invariance of Dürer's square might be projected onto the transposition arrows: let us for instance label T_1 the upper arrow and T_3 the lower one. All these choices allow us to univocally complete the K-nets for tetrads 1–4 (Fig. 6a). In doing so, the inversion operators become equally fixed and invariant for all K-nets, I_7 and I_{11}, respectively. In brief, our new K-nets have exactly the same arrows, which implies strong isography between them. As mentioned previously, we wish to preserve the square shape of the networks to

Fig. 6 (a) Strong isographies among the K-nets for tetrads 1–4 (b) The same for tetrads 5–8. Squared numbers in the centre of each K-net identify the embedded pitch-class and, above all, become representative of the whole K-net for larger hyper-K-nets. This style of notation applies to further figures in this essay
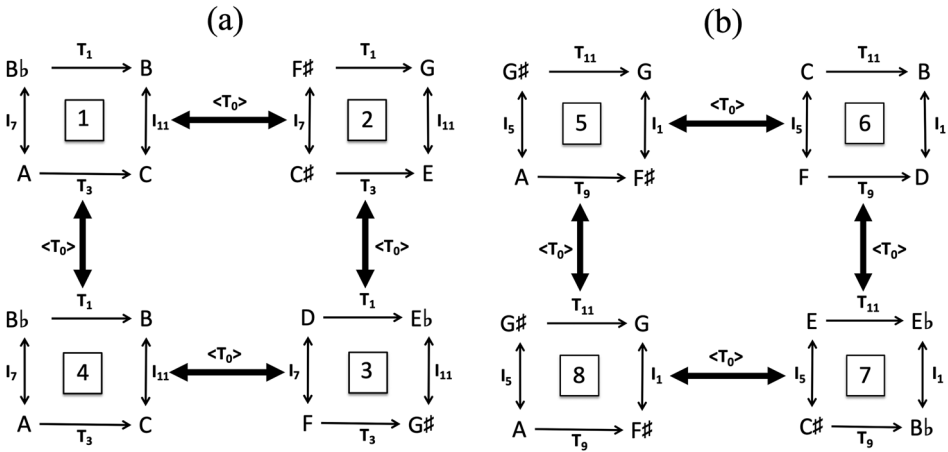

allow the geometrical unfolding of our analytic proposal; this fact explains our particular distribution of the $<T_0>$ arrows.

Despite the successful embedding of subtractive invariance in the transposition operators, it seems there is a problem with the inversion ones. Their indices in our K-nets are 7 and 11, which are not congruent with 15 and 19 modulo 12. Apparently, the additive invariance of Dürer's square has been violated. It is, however, an effect of having A as the axial – that is, referential – pitch class. Manoury's system of reference is equivalent to considering A = 1, B♭ = 2, B = 3, and so on. This is merely a transposition of the standard system where C = 0, C♯ = 1, D = 2, and so on. Let us consider transposition T_4, which has so far been left implicit, for transforming tetrad 1: (A, B♭, B, C) thus becomes (C♯, D, E♭, E). When the new pitch classes are projected onto the structure of the K-net of the former collection, that is, respecting the spatial distribution of arrows, transposition operators T_1 and T_3 remain, but dyads (C♯, D) and (E♭, E) allow inversion operators I_3 and I_7 to emerge. New indices 3 and 7 are this time congruent with 15 and 19 modulo 12. Consequently, the constants of the additive invariance of Dürer's square were effectively preserved, although hidden by Manoury's axis.

The next objective is to build the K-nets for the inverted tetrads, that is, 5–8. For that purpose, our strategy will be slightly different. We will start again by positioning, in a squared structure, the horizontal sides for the transposition arrows – once more from left to right – and the vertical ones for the inversion arrows. Next, as we did previously, we will preserve the spatial distribution of each I-related tetrad within its K-net: each I-related pitch class must occupy the

Fig. 7 Isographic cube connecting the different K-nets for tetrads 1–8
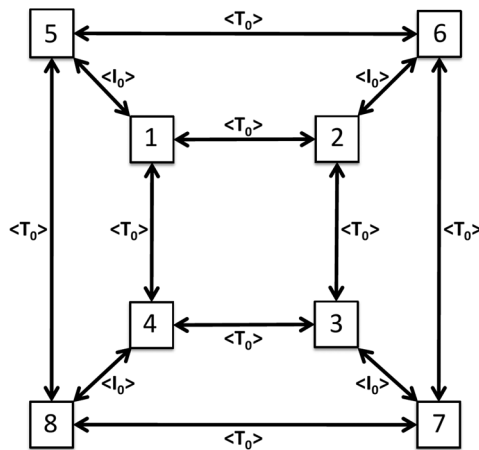
same position. Consider for instance the spatial distribution of the K-net for tetrad 1; in the K-net for tetrad 5, pitch class A remains invariant, whereas G♯ takes the place of B♭, G the place of B and F♯ the place of C. By doing the same with tetrads 6, 7 and 8, we obtain a diagram akin to that for tetrads 1, 2, 3 and 4 (Fig. 6b). The difference is in the indices: transposition operators are now T_{11} and T_9, whereas inversion operators are I_5 and I_1. Consequently, the circuit of strong isographies remains unaltered.

Comparing the K-nets for tetrads 1–4 with those for tetrads 5–8, it becomes evident that both families are related via negative isography $<I_0>$. This outcome is consistent with a theoretical result: given a set x and any K-net on it, if we apply some inversion I_a to x and replicate the structure of arrows among inverted pitch classes, the K-nets for x and $I_a(x)$ become related via the negative isography $<I_{2a}>$. In our case, we already mentioned that the A axis induces I_6-relations, which leads to negative isography $<I_0>$ when preserving the spatial structure of the K-nets. All in all, it is possible to merge both circuits of isographies in Fig. 6: the result – keeping in the spirit of the hypercubic metaphor – is a cube in which the vertices are the K-nets we have built, and its sides are only the isographies $<T_0>$ and $<I_0>$ (Fig. 7). Evidently, we might have erased the K-nets of tetrads 5 and 8, as they repeat those of tetrads 1 and 4; the cube would have become a triangular prism. We prefer, however, to retain the redundancy within the cubic structure as it will prove useful for subsequent analytical purposes.

It is time for a short recapitulation. Through this section, we have provided two analyses of the eight tetrads forming the opening of *Melencolia*. The first analysis was quite succinct and object-oriented. By means of Forte's set theory, we reduced the tetrads to a pair of prime forms and highlighted the importance of inversion. The second analysis was much longer and intricate; it adopted a transformational viewpoint and incorporated the particular properties of Dürer's

Fig. 8 Comparison of Manoury's *carré I* (Dürer), *carré II* and *carré III*

square. By using K-nets, we were able to represent the tetrads with only two kinds of networks and construct a hyper-K-net with only two types of isographies. Both perspectives have provided intriguing outcomes, but the reader may wonder whether the complexity of the second path was worth the effort. Our response is affirmative: further analyses of new pitch-class sets – equally obtained from Manoury's squared arrays – will demonstrate that the number of prime forms and their distribution may change without displaying a particular arrangement at first glance, whereas the transformational approach discloses many more patterned features.

Exploring Klumpenhouwer Tesseracts

Let us return to the combinatorial strategies which led to the actual composition of *Melencolia*. Manoury's sketches for the formal structure of the piece did not only incorporate Dürer's square but also progressive transformations of its numerical entries. In particular, Plate 2 contains three arrays (summarised in Fig. 8) which are labelled *carré I*, *carré II* and *carré III*, that is, squares I, II and III in French. A close look at these squares allows us, by projecting the patterned structure of black and white cells from Fig. 1, to deduce how they are related. Square I is simply Dürer's square. In Square II, numbers have been slightly altered: for each horizontally adjacent pair of entries sharing a colour in Dürer's square, the larger number is reduced by 1 whereas the smaller is increased by 1. For example, (16, 3) becomes (15, 4) and (2, 13) becomes (3, 12). This parallelism of increase and decrease is kept in Square III, where (15, 4) becomes (14, 5) and (2, 13) becomes (3, 12).

Although this mechanism is almost always repeated mechanically, the reader may wonder what happens, for instance, when a descending progression reaches value 1. To explain this and other situations, consider Plate 4. Here we find four arrays, each made of 4 rows and 23 columns. Let us consider the first column of each array and, preserving the order of the arrays, arrange the contents side by side into a square: we will have reconstructed Dürer's square. Applying the same protocol to the second column of each array restores Square II, and so on.

Manoury probably decided to split Dürer's square into its four columns in order to facilitate his work on the algorithmic deduction of the modified squares. The mechanism is quite simple and may be inferred from the upper array. As we have already seen, this array starts with the first column of Dürer's square. We

Plate 4 Manoury's arrays defining the progressive transformation of Dürer's square.
Reproduced by kind permission of the author
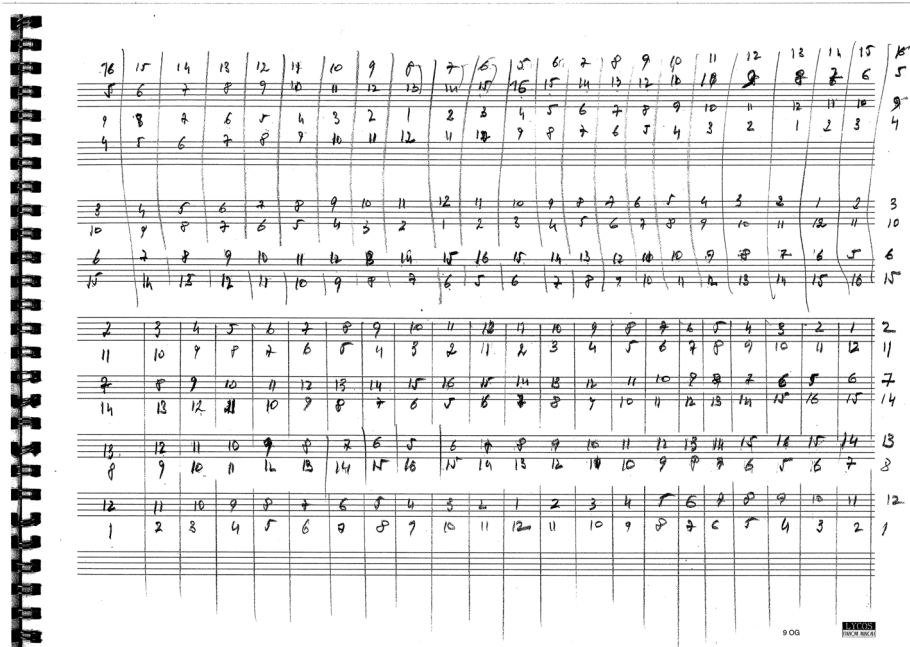
have already pointed out which values increase by 1 and which decrease by 1:
thus, column (16, 5, 9, 4) becomes (15, 6, 8, 5). The process may then take two
possible paths. When no entry in the most recently generated column reaches
the maximum or minimum values of Dürer's square – 1 or 16 – the ongoing
arithmetic progressions continue their course. Otherwise, a rebound takes place:
once a maximum or minimum value is reached, the arithmetic progression is
reversed. This reversal also takes place in the related row: if, for instance, the
maximum or minimum value was reached in the first – alternatively the fourth
– row, the second – alternatively the third – row would equally see its arithmetic
progression reversed, and vice versa. This associated change of direction serves
to offset the one which triggered the rebound. Let us continue our examination
of the upper array to illustrate our previous comment. Neither 1 nor 16 belongs
to the second column (15, 6, 8, 5), which means that the following will be (14, 7,
7, 6). Several steps later, the eighth column becomes (8, 13, 1, 12); as the third
entry is 1, the two lower rows reverse their respective arithmetic progressions,
which leads to (7, 14, 2, 11) as the next column. The process ends with the 23rd
iteration, when the first column of Dürer's square is restored.

There are resemblances between Manoury's sketch and the former's arrays,
such as the one we adapted in Fig. 4. Both Manoury's arrays and what we
already found in Perle's sum tetrachords lead us to examine which properties

of Dürer's square are preserved in the new squares generated by Manoury's algorithmic strategies. These are not necessarily magic squares because numbers are often repeated within them, as happens for instance in Squares II and III. In addition, there are a few squares in which certain properties of Dürer's square do not apply: these are the squares generated by the ninth, tenth, twenty-first and twenty-second columns of Manoury's arrays. The composer probably detected some irregularities – they mark the moments in which the rebounds of the arithmetic progressions are unsynchronised – and therefore withdrew them. Otherwise, the magic sum, additive invariance of rows and subtractive invariance are preserved. Again with regard to Fig. 8, the first and second properties appear as merely trivial consequences of the increasing and decreasing progressions of adjacent entries sharing a same colour, which balance the sums of these entries.[6] It is interesting that, after the first sequence of rebounds, white and black adjacent pairs swap their sum. They swap back again for the last column, that is, when Dürer's square is reconstructed. Moreover, Manoury's squares preserve the subtractive invariance of rows because the arithmetic progressions run in parallel, except during the unsynchronised rebounds.[7]

There are nineteen passages in *Melencolia* which depend on thirteen different squares resulting from Manoury's procedure (for a comprehensive list of these passages, see Table 2). He used these squares in a way similar to what he did with Dürer's square: he selected a pitch class for each of his twelve new squares which served as an axis for the derivation of pitch-class sets. The choice of the twelve new axes was not arbitrary. They correspond to the pitches which Manoury already obtained from the first three rows of Dürer's square. These pitches can be retrieved from tetrads 1, 2 and 3 in Fig. 5, respecting the precedence of pitch classes inferred from reading Dürer's square from left to right, and from top to bottom. The resulting list of axes coincides with the following dodecaphonic row (C, B, B♭, A, C♯, F♯, G, E, F, D, E♭, A♭). Thus, to any square n with n greater than 1, Manoury attached the $n - 1$ pitch class of the twelve-note row as its axis. Because we do not have enough space to provide an extensive tour of all the passages in *Melencolia* where the twelve new squares are called upon, we will only focus on a few particular situations.

Let us start with Square V (Fig. 9b), to which the A axis is attached, as was already the case at the beginning of the piece with Dürer's square. Both elements allow us to calculate eight new tetrads as in the previous case. We map the numbers in each column of the square into ascending semitones in order to obtain tetrads 1_V–4_V – reminding ourselves to subtract one semitone because the axial pitch class A already stands for value 1. For instance, the first row of Square V (12, 7, 6, 9) generates tetrad 1_V (G♯, E♭, D, F). Tetrads 5_V–8_V, as the A axis induces inversion I_6, can be immediately derived from 1_V–4_V. At the end of this process, eight tetrads have been deduced (Fig. 9a).

These pitch-class sets are found in bars 161–170. Bars 161–162 (Ex. 2) unfold two kinds of figures: short accentuated notes – mainly harmonics – displaying a descending movement, on one hand; on the other, different kinds of

Fig. 9 (a) Different tetrads derived from Manoury's Square V (b) Square V (c) Strong isographies among the K-nets for tetrads 1_V–4_V

Ex. 2 *Melencolia*, bars 161–162. © Durand Editions, 2013. Reproduced by kind permission of Universal Music Publishing and Durand Editions

Table 2 Synoptic description of all the passages in *Melencolia* based on the magic square category. We keep Roman numerals and indices for square numbers and rehearsal letters as they are found in Manoury's sketches. The bracketed rehearsal letter does not appear in the score, although it is found in his drafts. Bar numbers do not reproduce sketch data but actual values within the score. Asterisks highlight the presence of the axis within the score

Bars	Rehearsal	Square	Axis	Square texture	Overlapping texture
1–8	I_A	I	A*	Static	None
9–16	I_B	II	C		*Tremoli* and *pizz.*
17–24	I_C	I	A	Short *tremoli*	None
25–32	I_D	II	B	Static	*Tremoli* and *pizz.*
33–37	I_E	I	A	Short *tremoli* and *pizz.*	None
38–41	I_F	IV	B♭*	Extreme voices	Central voices
161–170	IV_A	V	A	Static and *pizz.*	Scales
171–186	IV_B			Static, repeated, and *pizz.*	Grammars (varia)
187–190	IV_C			Static and *pizz.*	Scales
293–308	VI_C	VI	C♯*		None
324–341	VI_E	VII	F♯*	Repeated and *pizz.*	
376–393	VII_B	VIII	G	Static and *pizz.*	*Pizz.* double strings
394–413	VII_C	IX	E	Static	Scales
437–458	VII_E	X	F*	Hocket	None
459–475		XI	D		
503–408	$VIII_B$	XII	E♭*	Short melodic accents	
539–548	$[VIII_D]$	XIII	A♭*	Repeated and accents	
577–581	$VIII_H$	I	A	Static	
582–587	IX_A	III	C		Viola phrases
588–597	IX_B	II	B		Cello phrases
598–606	IX_C	I	A*		None

longer sounds with a resonant effect, namely ***fp*** sustained harmonics, left hand *pizzicati* whereas the screw of the bow frog is placed on the string for a metallic effect, and sounds produced by the crotales. Manoury used the pitch classes derived from Square V for the resonant figures: the pitch classes of tetrad 1_V – or 2_V, which is identical to it – are distributed across both violins whereas those of tetrad 8_V – or 7_V – are placed in the viola and the cello.

Unlike the tetrads derived from Dürer's square, all collections from 1_V to 8_V can be reduced to prime form 4–13 [0, 1, 3, 6]. The repeated rows in Square V also give rise to a larger repetition of pitch-class sets. Forte's analytical perspective therefore seems – at least at first glance – to fall short of unveiling a deeper formal connection between the tetrads generated from different squares, whereas K-nets elegantly solve this issue.

We may draw K-nets for tetrads 1_V–4_V by following exactly the same protocol as we described earlier for tetrads 1–4. We can opt for this path – and here we are emphasising a fundamental aspect of our analytic proposal – because Square V preserves the additive and subtractive invariances of Dürer's square. The result is a circuit of strong isographies among the four K-nets (Fig. 9c)

Fig. 10 K-tesseract connecting all the K-nets for tetrads 1–8 and 1_V–8_V
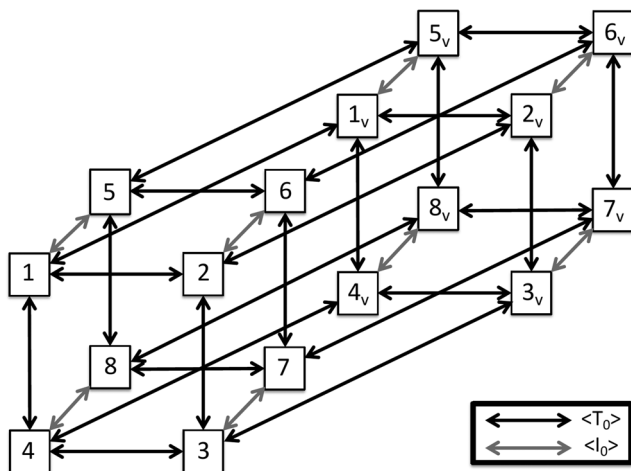
which, structurally speaking, is exactly the same as that presented in Fig. 6a. Analogously, we may draw a circuit of K-nets for tetrads 5_V–8_V and retrieve the same structure of K-nets and circuit of strong isographies as that displayed in Fig. 6b for tetrads 5–8. Continuing the analogy, as both squared circuits are preserved and the A pitch-class axis is shared between Dürer's square and Square V, we may draw an isographic cube for tetrads 1_V–8_V and retrieve the same structural isographies as those shown in Fig. 7 for tetrads 1–8. Finally, as for any value α from 1 to 8, K-nets for tetrad α and α_V have the same distribution of arrows, the respective pair of K-nets can be linked by a strong isography. All these results lead us to build a tesseract of isographies for connecting all the K-nets for tetrads 1–8 and 1_V–8_V, that is, connecting both isographic cubes (Fig. 10). In doing so, we expand our hypercubic metaphor. We will henceforth use the term K-tesseract for such transformational layouts. This particular K-tesseract only displays isographies $<T_0>$ and $<I_0>$.

K-tesseracts become more complex when the pitch-class axes are moved because the underlying K-nets also vary. For instance, consider section VII_E of *Melencolia*, which is based on Manoury's Square X and on the F axis (Fig. 11b). We must consider this new axis for calculating the tetrads. Now, F stands for value 1 when calculating pitch classes: for instance, the first row of Square X (5, 10, 11, 8) becomes tetrad 1_X (A, D, E♭, C). As F is the axis, it also induces inversion I_{10} instead of I_6 in the previous cases. Thus, the I_{10}-related tetrad of 1_X is 5_X (C♯, A♭, G, B♭). Other tetrads originated by Square X are obtained in the same way (Fig. 11a).

These sets are found at bars 437–458. In bars 438–440 (Ex. 3), the viola periodically plays F_6 on a crotale – matching with the axis – whereas the violins and the cello unfold a string hocket based on the tetrads in question. For

Fig. 11 (a) Different tetrads derived from Manoury's Square X (b) Square X (c) Strong isographies among the K-nets for tetrads 1_X–4_X

Ex. 3 *Melencolia*, bars 438–440. © Durand Editions, 2013. Reproduced by kind permission of Universal Music Publishing and Durand Editions
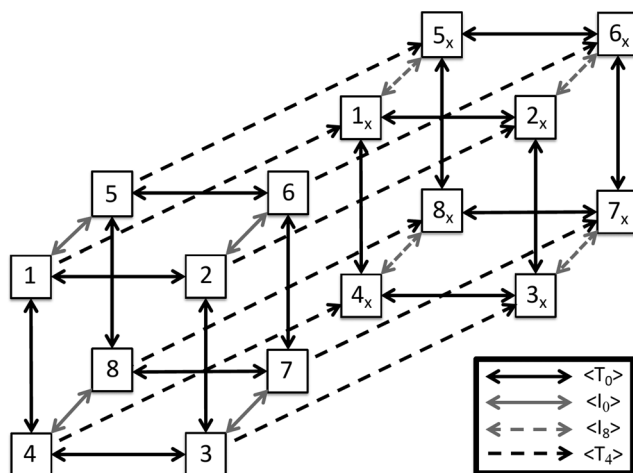
Fig. 12 K-tesseract connecting all the K-nets for tetrads 1–8 and 1_X–8_X

instance, there are eight notes played by the strings in bar 438: the notes in violin I and the D_7 in violin II match with tetrad 1_X; in turn, the notes in the cello and the C_6 in violin II match with 7_X except for a missing A. It can be surmised that Manoury or his publisher mistranscribed a B_5 in the cello, because this pitch appears twice. Similarly, we find tetrads 2_X and 6_X – or 4_X and 8_X, which are the same – at bar 440 except for a missing G this time.

From the perspective of set theory, we discover a different distribution from those related to Dürer's square and Square V. This time, tetrads 1_X, 3_X, 5_X and 7_X are T/I-related to prime form 4–13 [0, 1, 3, 6], whereas tetrads 2_X, 4_X, 6_X and 8_X are T/I-related to prime form 4–13 [0, 1, 2, 3].

Let us draw K-nets for tetrads 1_X–4_X by following the same protocol as in previous cases (Fig. 11c). Compared with Figs. 6a and 9c, transposition operators and strong isographies are preserved. The indices for the inversion operators have changed however: they are now 11 and 3 because of the displacement of the axial pitch class. In addition, as tetrads 1_X–4_X are related to tetrads 5_X–8_X by means of inversion I_{10}, we may generate structurally matching K-nets for the last sequence of tetrads. They are automatically related – reminding us of a theoretical result enounced above – to the former sequence via negative isography $<I_8>$. Therefore, we obtain a cubic net of isographies connecting the K-nets of the tetrads from Square X which we now want to place in front of the cube built from Dürer's square in Fig. 7 in order to 'close' the K-tesseract. Comparing Figs. 6a and 11c, it is evident that, for any value β from 1 to 4, K-nets for tetrad β and β_X are related via isography $<T_4>$. Applying laws for isographic composition (Klumpenhouwer 1998), this result can be extended to values β from 5 to 8, and the K-tesseract is finally settled (Fig. 12).

Fig. 13 (a) Square XIII. (b) Isographically related K-nets for triads 1_{XIII} and 5_{XIII}
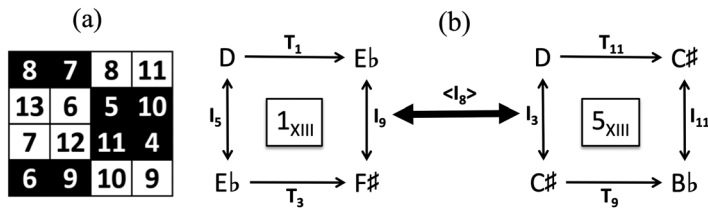
A comparison between this K-tesseract and the previous tesseract in Fig. 10 reveals two salient differences. First, the negative isographies connecting the opposite sides of each main cube in the new tesseract are different. Second, the isographies connecting both main cubes are no longer $<T_0>$ but $<T_4>$. Both features are a consequence of the axis displacement. As we explained earlier, this displacement can be regarded as a factor of transposition of the tetrads induced by the axes. In this sense, let us remember a theoretical result similar to the one we already enounced apropos of inversions: given a set x and any K-net on it, if we apply a transposition T_b to x and replicate the structure of arrows among transposed pitch classes, the K-nets for x and $T_b(x)$ become related via the isography $<T_{2b}>$. In this case, pitch classes are not transposed, but although additive and subtractive invariances are preserved, the result still applies. It is in fact consistent with the construction of our second K-tesseract: as the axis moved from A to F, transposition T_8 is implicit, which leads therefore to $<T_4>$ among the equivalent tetrads of different squares.

Rhythmic Excursus

In addition to using arrays to produce sets, Manoury took advantage of his arithmetical procedures in very sparse situations for developing rhythmic patterns. This is the case, for instance, of section $VIII_D$, which spans bars 539–548 and is related to Square XIII (Fig. 13a). In its opening bars (Ex. 4), both violins and the viola unfold a continuous flux of demisemiquavers emphasised by salient accents; the crotale – controlled by the cellist – enters when the accents of the three string instruments converge. Both violin II and the crotale only play $G\sharp_6$, which matches with the pitch-class axis of Square XIII. The centrality of this axis, as happened with the opening bars of *Melencolia*, is observed in terms of register, something which is aurally recognisable. The first row of this square consists of (8, 7, 8, 11). By following the same process as in previous examples, we obtain the melodic four-note sequences ($E\flat_7$, D_7, $E\flat_7$, $F\sharp_7$) and – reading from right to left – ($B\flat_5$, $C\sharp_6$, D_6, $C\sharp_6$). These sequences are found in violin I and in the alto, respectively.

Notice that, this time, we have used the term melodic sequences instead of tetrads. The reason is manifest: because both sequences contain repeated

Ex. 4 *Melencolia*, bars 539–542. © Durand Editions, 2013. Reproduced by kind permission of Universal Music Publishing and Durand Editions

pitches, their related pitch-class sets are not tetrads but triads – which we label, in accordance with the terminology employed throughout our article, as 1_{XIII} and 5_{XIII}. In particular, both triads belong to Forte's prime form 3–3 [0, 1, 4]. In parallel, we may draw their respective K-nets with the same rules we developed for previous tetrads (Fig. 13b) – incorporating for this purpose pitch-class repetition – and verify that the transformational structure is still preserved.

The main purpose of our excursus, however, was to analyse rhythm. Counting the number of demisemiquavers between the accents in both violins and the viola generates the numeric sequences (8, 13, 7, 6) for violin I and the viola, and (8, 7, 8, 11) for violin II, which correspond to the first column and the first row of

Square XIII, respectively. The same happens throughout the section in relation to further rows and columns of the same square. Consequently, Manoury used his squared array towards a personal interpretation of Olivier Messiaen's chromatic durations.[8] A particular feature emerges from this compositional choice. The accents in violin I – alternatively the viola – and violin II start together but finally become unsynchronised; after 34 demisemiquavers, they converge at bar 541, and so on. This is an outcome of the preservation of the magic sum across Manoury's modifications of Dürer's square.

Analytical Abstractions: Summary and Extension

Throughout the different passages of *Melencolia* that we have analysed, several more or less patterned structures have emerged. Although the use of pitch-class sets derived from mathematical squares is not pervasive through all sections of Manoury's quartet, their recurrent use – and particularly some symmetries which can be found in Table 2 – is probably the main feature of the piece's underlying coherence. Interestingly, as the examples throughout this essay have corroborated, the composer contextualised these pitch-class sets in many different ways, which enabled him to achieve a subtle balance between surface diversity and formal unity.

Thanks to set theory, we identified that Dürer's square gives rise to two categories of T/I-related tetrads with underlying prime forms 4–1 [0, 1, 2, 3] and 4–13 [0, 1, 3, 6]. The same sets could be found within Square X; yet, the connections between analogous tetrads in both squares remained unclear to us provided that we remained focused on the basic tenets of set theory. Moreover, Square V only generated T/I-related transformations of prime form 4–13 [0, 1, 3, 6]. Things seemed to go wrong even with Square XIII, which generated T/I-related triads with underlying prime form 3–3 [0, 1, 4], thus implying some kind of pitch-class 'collapse' caused by repetition. Neither inclusion nor the reciprocal complement relationship are found between Forte's prime forms 3–3 [0, 1, 4] and 4–1 [0, 1, 2, 3] or 4–13 [0, 1, 3, 6]. In short, pitch-class set theory has not been sufficient for a straightforward explanation of some deep relationships which connect the tetrads – or ultimately the triads – generated from Manoury's disparate squared arrays.

In contrast, K-nets have proven particularly suitable for this purpose. It is time therefore to initiate a more abstract discussion of our model of K-tesseracts (Fig. 14). Let A and B stand for any two distinct squares, and α and β represent the respective pitch-class axes for each of these squares. When A or B equals 1, then Dürer's square is implied. We can spot three different levels of isographies within the K-tesseract. First, K-net circuits derived from local readings of each square – that is, for sets 1_A–4_A, 5_A–8_A, 1_B–4_B and 5_B–8_B – are connected by strong isographies. Second, inversionally related sets – for example 1_A and 5_A – may preserve the original display of arrows of their K-nets. In this case, these

Fig. 14 Generalisation of K-tesseracts relating the pitch-class sets derived from any pair of squares
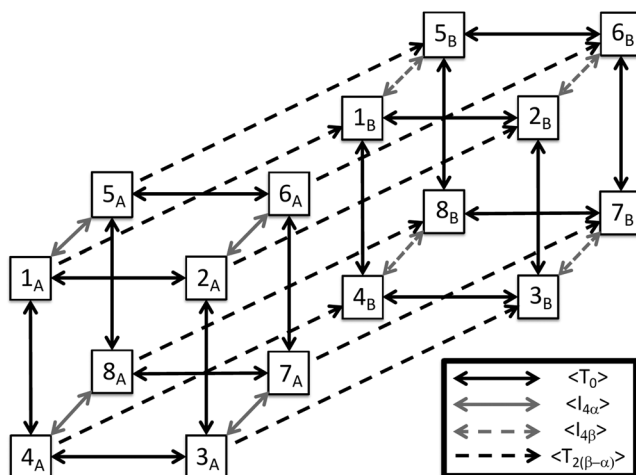
nets will be related by a negative isography which, depending on the symmetry axes, can be either $<I_{4\alpha}>$ or $<I_{4\beta}>$. Finally, the K-nets of analogous sets from different squares – that is, e_A and e_B where e can take any value from 1 to 8 – are related by $<T_{2(\beta-\alpha)}>$, which captures the axial displacement. In a case where there is no change of axis, this isography is trivially strong. These three isographic levels are distributed in a patterned way from a geometric perspective. Strong isographies are the edges of the four parallel faces of the K-tesseract and therefore define its vertical and horizontal dimensions. Negative isographies define the sagittal dimension and, together with the strong isographies, make up two facing cubes which relate to each of Manoury's Square, respectively. Positive isographies connect both cubes, resulting therefore in an additional fourth dimension which gives rise to the K-tesseract. It is important to highlight, as one of our major analytical results, that these isographic levels are a consequence of the additive and subtractive invariances of Manoury's Square and/or their respectively fixed pitch-class axes (as summarised in Table 3).

All the K-nets we have shown for sets 1_A–4_A have arrows T_1 and T_3. The K-nets of their I-related sets therefore exhibit – because of the negative isographies – arrows T_{11} and T_9. The latter arrows can of course be reversed to yield T_1 and T_3 again. This choice demands that transposition operators become invariant in all the K-nets of our model. Conversely, only inversion operators vary according to the pitch-class axis. By means of several of the results obtained previously, it is possible to calculate all the inversion operators which take part in our K-nets for each axis, for any set as well as its axial inversion (Table 4). A meaningful inference which we can make through this table is that, among all the possible K-nets of our analytical model, there are only six structural configurations. These

Table 3 Summary of the different isographies conforming a K-tesseract and their matching mathematical properties and/or compositional choices

Dimension	Isography	Cause
Horizontal (1st)	$<T_0>$	Additive and subtractive invariance in any of Manoury's squares.
Vertical (2nd)		
Sagittal (3rd)	$<I_{4\alpha}>$	Pitch-class axis α.
4th Dimension	$<T_{2(\beta-\alpha)}>$	Additive and subtractive invariance in any of Manoury's squares + Displacement from pitch-class axis α to β.

Table 4 Summary of inversion arrows for any K-net in our model and for each pitch-class axis. The 'Isography' column highlights the link between K-nets of two I-related sets by means of the mirroring axis. The shaded cell matches the UrK-net

Pitch-class axis	I arrows (original)	Isography	I arrows (inversion)
C or F♯	I_1 and I_5	$<I_0>$	I_7 and I_{11}
C♯ or G	I_3 and I_7	$<I_4>$	I_9 and I_1
D or A♭	I_5 and I_9	$<I_8>$	I_{11} and I_3
E♭ or A	I_7 and I_{11}	$<I_0>$	I_1 and I_5
E or B♭	I_9 and I_1	$<I_4>$	I_3 and I_7
F or B	I_{11} and I_3	$<I_8>$	I_5 and I_9

configurations always preserve arrows T_1 and T_3 but let their inversion operators vary between six possibilities: I_1 and I_5, I_3 and I_7, I_5 and I_9, I_7 and I_{11}, I_9 and I_1, I_{11} and I_3. One of these configurations was already evoked in our first analysis. It fixes arrows T_1, T_3, I_3 and I_7, thus capturing the true constants – modulo 12 – of the additive and subtractive invariances in Dürer's square. Coining a neologism inspired by the Schenkerian tradition, we henceforth define this archetypal structure as the *UrK-net*.

As there are only six K-nets in our model, we can calculate the 72 pitch-class sets from which they may be constructed. For each K-net, it is merely necessary to select a pitch class – for instance, in the lower-left corner – and fill all the others by following the arrows. Next, it is possible to reduce the variety of sets by considering T/I-relationships: the outcome is a heavily patterned map (Table 5). There are, consequently, 12 sets with prime form 3–3 [0, 1, 4], 24 sets with prime form 4–1 [0, 1, 2, 3], 24 sets with prime form 4–13 [0, 1, 3, 6] and 12 sets with prime form 4–20 [0, 1, 5, 8]. All of these forms were corroborated by our previous analyses, save the last prime form for which we did not encounter any corresponding set. There were, however, representatives of this form in Ex. 4. The melodic four-note sequences of violin I and of the

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Table 5 Map of pitch-class sets that could be generated by all possible K-nets in our model. Reminder: 3–3 [0, 1, 4], 4–1 [0, 1, 2, 3], 4–13 [0, 1, 3, 6] and 4–20 [0, 1, 5, 8]

| Lower-left pitch-class	Inversion operators of K-nets					
	I_1 and I_5	I_3 and I_7	I_5 and I_9	I_7 and I_{11}	I_9 and I_1	I_{11} and I_3
C or F♯	4–1	3–3	4–13	4–20	4–13	3–3
C♯ or G	3–3	4–1	3–3	4–13	4–20	4–13
D or A♭	4–13	3–3	4–1	3–3	4–13	4–20
E♭ or A	4–20	4–13	3–3	4–1	3–3	4–13
E or B♭	4–13	4–20	4–13	3–3	4–1	3–3
F or B	3–3	4–13	4–20	4–13	3–3	4–1

Fig. 15 Embedding of the UrK-net onto a circular diagram elucidating, by means of four representatives, the invariant structure of any possible pitch-class set in our model. Arrows stand for transposition operators while dashed lines stand for inversion ones

4-1 **3-3** **4-13** **4-20**
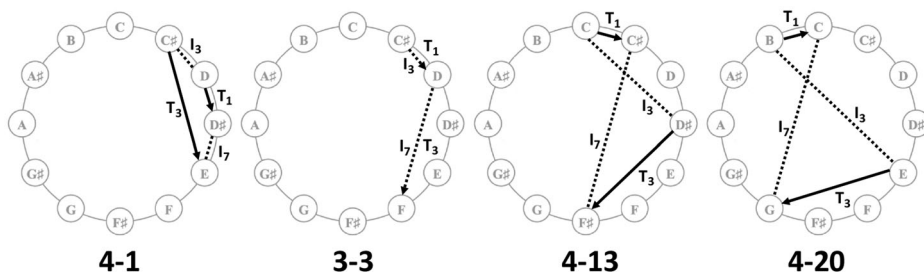
viola in bars 541–542 are (A♭$_7$, D♭$_7$, C$_7$, F$_7$) and (B$_5$, E$_6$, E♭$_6$, A♭$_5$), respectively, which match with Forte's prime form 4–20 [0, 1, 5, 8]. These tetrads – 2_{XIII} and 6_{XIII} – result from interpreting the second column of Square XIII in Fig. 13. Our transformational model has thus been able to predict a category of T/I-related pitch classes which had gone unnoticed throughout our previous analyses.

As a final proof of the advantages of K-nets, we may embed onto circular diagrams four representatives among the possible pitch classes generated by the UrK-net of Dürer's square (Fig. 15). The resulting diagrams are quite eloquent from an analytical perspective. First, they demonstrate that the structural equivalences among the four types of sets were not self-evident; they have emerged through the K-nets. Second, the representatives of Forte's classes 4–1 and 4–20 are invariant under inversions I_5 and I_{11}, respectively. This feature implicitly explains why, on the map in Table 5, the total number of occurrences of these classes is only half of that of classes 3–3 and 4–13. As can be seen from

Fig. 16 (a) Underlying UrK-net of a generic gnomon 4-square (b) Dürer's square and its UrK-Net (c) *Quadraginta* square and its UrK-net
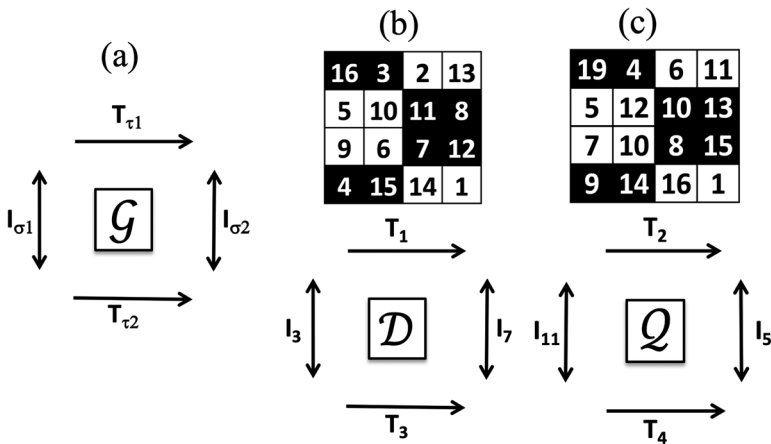

the representatives chosen in this figure, the latter classes have no symmetry axis. Third, the pitch-class 'collapse' from tetrad to triad can be visualised in the fact that inversion and transposition operators become superposed onto shared lines. In this example, the repeated pitch class is D.

Compositional Prospects

Manoury's squares are not, generally, magic squares. Nevertheless, as both additive and subtractive invariances are preserved, the UrK-net remains referential for them. This fact is pivotal to our argumentation: it is not the property of being magic, but the invariances which determine the structure of the UrK-net. Let us therefore reach a new level of abstraction, abandoning Dürer's square in favour of a generic square which must retain at least the invariance properties – that is, a gnomon square. On one hand, the arrows of its UrK-net are defined by four numerical values, namely τ_1, τ_2, σ_1 and σ_2 (Fig. 16a): τ_1 and τ_2 project the subtractive invariances onto the transposition operators, whereas σ_1 and σ_2 project the additive invariances onto the inversion operators. However, these values are not independent, as $\tau_2 = |\sigma_1 - \sigma_2| - \tau_1$.

The main interest of this abstraction is not analytic but intrinsically compositional. In fact, given a set of reasonable values for a magic sum and constants τ_1, τ_2, σ_1 and σ_2, it is possible to create new gnomon squares beyond Dürer's one (Fig. 16b). To illustrate the potentialities of this construction, let us define the *Quadraginta* square (Fig. 16c): here, the magic sum is 40, the constants of the additive invariances are 23 and 17 – which leads to indices 11

Fig. 17 Strong isographies among K-nets of four different tetrads derived from the *Quadraginta* square, analogous to those we built for analysing *Melencolia*
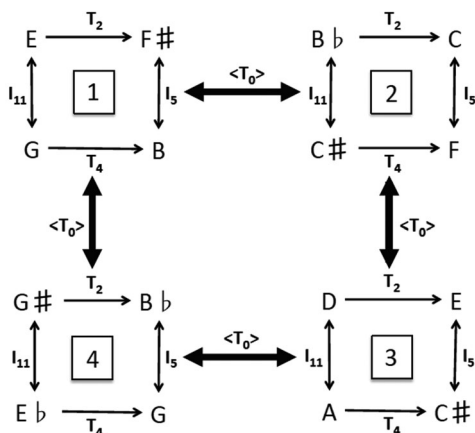
and 5 for the inversion arrows – whereas the constants of subtractive invariances are 2 and 4. If, in a way similar to Dürer's square, we fix the maximum and minimum values – that is, 1 and 19 – as the entries in the skewing cells at the upper-left and lower-right corners; all other entries will be automatically filled thanks to the invariance properties. Notice that the *Quadraginta* square preserves the magic sum by rows and columns but is not magic because value 10 appears in two different cells. Once the square and its UrK-net are fixed, it is possible to use them to generate tetrads – eventually smaller pitch-class sets – as Manoury did in *Melencolia*. These tetrads are automatically related by strong isographies (Fig. 17) and can therefore be considered from a compositional perspective.

This brings us back to Perle's theoretical constructions. While discussing his sum tetrachords, we found a K-net structure which remained invariant through all the sequence. We could also have continued the chain of strong isographies in Fig. 3 for all its chords. The sequence is made up of twelve chords because of the underlying cycles of perfect fifths which Perle used to generate it. As the invariant K-net we built for the analysis of Perle's example could be regarded as an UrK-net among many potential ones, a significant consequence arises. Let M represent any of Manoury's squares: collections 1_M–4_M – alternatively 5_M–8_M – might be considered, in terms of pitch classes, as a subset – perhaps with repetitions – of the sequence of Perle's sum tetrachords which shares the same structural K-net. For this reason, we henceforth label sets 1_M–4_M – alternatively 5_M–8_M – as *Perle-Manoury collections*.

Further Analytical Abstractions: On Recursion and Movement

Let us imagine that we have the K-tesseract relating the Perle–Manoury collections from Dürer's square to those from Manoury's Square II, as well as the K-tesseract for Squares II and III. We may combine these tesseracts by means of their shared cube and consequently obtain a larger hyperprism: this operation is similar to gluing together two cubes by opposing a couple of equivalent faces. The process can be iterated – adding the K-tesseract for Manoury's Squares III and IV, and so on – until the cube of K-nets for Square XIII is reached. The resulting structure would form a long chain of twelve aligned K-tesseracts, made of 104 vertices and 252 isographies.[9] All the vertices belong to the isographic variants of the UrK-net previously summarised in Table 4. The isographies are also neatly distributed: on one hand, 104 strong isographies – those for the vertical and horizontal dimensions – configure a large ensemble of 26 parallel faces; on the other hand, 52 negative isographies – for the sagittal dimension – and 96 positive ones[10] – for the fourth dimension – are determined by the aforementioned dodecaphonic row (C, B, B♭, A, C♯, F♯, G, E, F, D, E♭, A♭) which Manoury generated from Dürer's square for obtaining an ordered sequence of pitch axes, preceded by an initial A. In particular, the negative isographies – in packages of four – reflect, multiplied by 4, the numerical structure which lies behind the tridecaphonic row: $<I_0>$, $<I_0>$, $<I_8>$, $<I_4>$, $<I_0>$, $<I_4>$, $<I_0>$, $<I_4>$, $<I_4>$, $<I_8>$, $<I_8>$, $<I_0>$ and $<I_8>$. Similarly, the positive isographies – in packages of eight – reflect, multiplied by 2, the intervals between contiguous pitches of the other thirteen-tone row: $<T_6>$, $<T_{10}>$, $<T_{10}>$, $<T_{10}>$, $<T_8>$, $<T_{10}>$, $<T_2>$, $<T_6>$, $<T_2>$, $<T_6>$, $<T_2>$ and $<T_{10}>$.

The choice, twice in the above paragraph, of using the term packages was not innocent. The huge hyperprism which we have defined ideally embodies our desired hypercubic metaphor, but it is impractical visually speaking. Recursion – that is, packaging isographies with some associated hierarchy – may facilitate a visualisation of the whole transformational process. Our schematic model of recursion is inspired by Klumpenhouwer's work on Donald Martino (Klumpenhouwer 1991), although other notations are possible (e.g. Tymoczko 2007). Our choice, throughout the previous analysis, of using squared labels standing for K-nets in cubes and tesseracts was already a manner of recursion. We may keep the aforementioned 26 faces, because they are all squares made of strong isographies among K-nets within a Perle–Manoury collection. Next, each package of four negative isographies $<I_x>$ between K-nets of I-related Perle–Manoury collections may merge into a single hyper-hyper-arrow $<<I_x>>$. Analogously, each package of eight isographies $<T_y>$ connecting cubes may merge into a single hyper-hyper-hyper-arrow $<<<T_y>>>$. Carrying on this way over our hyperprism of isographies, the diagram – baptised *Perle-Manoury K-necklace* – becomes much more readable (Fig. 18).

Fig. 18 Perle–Manoury K-necklace for *Melencolia*. The dashed arrow simply closes the circuit

The term 'necklace' has mathematical resonances with combinatorics (Gilbert and Riordan 1961); yet, by selecting it, we are also leaving aside another mathematical metaphor, which is the hypercubic one. The Perle–Manoury K-necklace induces, visually speaking, the displacement of a hierarchised object through a path of hyper-hyper-hyper-arrows. It resembles the manoeuvre of a TIE-fighter spacecraft of the Empire in *Star Wars*. Our final diagram, consequently, embraces the 'motion' metaphor, which is often behind the transformational perspective in music theory (Attas 2009).

Afterthought: Into the Debate on K-nets Promiscuity?

Our Perle–Manoury K-necklace is among the densest analytical diagrams to have been constructed so far based on Klumpenhouwer's theory of an existing musical score. This huge density may bring to mind the quarrel on promiscuity around K-nets, with, on one hand, a scholar denouncing this particular feature (Buchler 2007, § 32–52) and on the other, those embracing its potential (e.g. O'Donnell 2007).[11] Our position somehow lies outside this discussion because, although our final diagram is truly dense, it is made of a few and strongly patterned elements, and it is, above all, derived directly from Manoury's squares and axes. In that sense, we have demonstrated that even negative isographies,

which have sometimes been considered a questionable tool in terms of musical intuition (e.g. Straus as cited in Losada 2007, § 17), can match with actual compositional intentions. Our aim throughout our whole essay has therefore aligned with the notion of 'contextual transformations' (Lambert 2000) in a quest for grasping, and further formalising, some of Manoury's compositional strategies.

NOTES

1. A second formal sketch by Manoury also incorporates the mention of a section categorised as a '*Lamento* in memoriam Emmanuel Nunes' in tribute to the Portuguese composer, who died as Manoury was writing his score.

2. Other composers with similar grammar-based approaches are Hanspeter Kyburz (Supper 2001, pp. 51–2), Enno Poppe (Knipper 2014 and 2017) and Alberto Posadas (Besada 2017, pp. 159–82 and 2019).

3. This property is, as a matter of fact, constructive. Let us fix the upper left pair of black entries. Owing to the magic sum, the sum for the white lower left pair is fixed equally. Next, and owing to the skewing cell properties, the sums for both the upper and lower right pairs are also fixed. Finally, and as a consequence of the structure, 2-subsquares in the gnomon square, the remaining four pairs for the central columns are fixed equally.

4. In the formulas, the straight brackets stand for absolute values, that is, those without regard to their respective sign.

5. In particular, transpositions reorder the position of the rows in the former Latin 12-square, whereas inversion I_0 exchanges rows and columns in the same way mathematicians talk about transposed matrices.

6. The additive invariance and the magic sum also hold for columns; it can be checked by recolouring the sequence of squares vertically. Nevertheless, the unsynchronised rebounds are situated elsewhere.

7. Again, the subtractive invariance also holds for columns, with different unsynchronised rebounds.

8. The technique is based on the additive principles Messiaen borrowed from Indian music and adapted to a Western serial context (Simundza 1987 and Hook 1998).

9. The number of vertices is 13 times the 8 tetrads related to each square. For the number of isographies, we have to avoid repetition while counting: 32 isographies for the starting K-tesseracts but next, when gluing the 12 new ones, only 20 for each.

10. Again, these numbers deserve some comment. The strong isographies are 4 times the 26 faces involved in which related tetrads are grouped. The negative isographies are 4 times the 13 different gnomon squares. The regular positive isographies are 8 times the 12 connections between pairs of cubes that give rise to K-tesseracts.

11. See also other objections against Buchler's claim by other authors already cited in our article (e.g. Klumpenhouwer 2007 and Stoecker 2007).

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ABSTRACT

Many Western art music composers have taken advantage of tabulated data for nourishing their creative practices, particularly since the early twentieth century. The arrival of atonality and serial techniques was crucial to this shift. Among the authors dealing with these kinds of tables, some have considered the singular mathematical properties of magic squares. This paper focuses on a particular case study in this sense: Philippe Manoury's Third String Quartet, entitled

Melencolia. We mainly analyse mainly several strategies conceived by the French composer – through his own sketches – in order to manipulate pitches and pitch-classes over time. For that purpose, we take advantage of Klumpenhouwer networks as a way to settle wide and dense isographic relationships. Our hyper-K-nets sometimes reach a total of 32 arrows that allow geometrical arrangements as tesseracts in which their different dimensions cluster related families of isographies. In doing so, we aim to provide an instructive example of how to contextualise K-nets and isographies as powerful tools for the analysis of compositional practices.